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On highest-energy state in the $\mathfrak{su}(1|1)$ sector of $\mathcal{N}=4$ super Yang-Mills theory

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Abstract: We consider the highest-energy state in the $\mathfrak{su}(1|1)$ sector of $\mathcal{N}=4$ super Yang-Mills theory containing operators of the form $\mathrm{tr}(Z^{L-M}\psi^M)$ where Z is a complex scalar and ψ is a component of gaugino. We show that this state corresponds to the operator $\mathrm{tr}(\psi^L)$ and can be viewed as an analogue of the antiferromagnetic state in the $\mathfrak{su}(2)$ sector. We find perturbative expansions of the energy of this state in both weak and strong 't Hooft coupling regimes using asymptotic gauge theory Bethe ansatz equations. We also discuss a possible analog of this state in the conjectured string Bethe ansatz equations.

Keywords: Bethe Ansatz, AdS-CFT Correspondence.

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Contents

1. Introduction

Recent advances in understanding of the AdS/CFT duality rely on various kinds of evidence that both the large N maximally supersymmetric Yang-Mills theory [1, 2] and the dual classical $\mathrm{AdS}_5 \times \mathrm{S}^5$ string theory [3, 4] are integrable models. An important part of the problem of solving the conformal large N SYM (or dual string theory) would be to compute the spectrum of anomalous dimensions (or string energies) as explicit functions of the 't Hooft coupling constant λ. These functions should describe a smooth interpolation from small λ (perturbative gauge theory) region to large λ (perturbative string theory) region.

Apart from a trivial case of BPS operators whose conformal dimensions are protected and, therefore, λ-independent, so far we do not know any other operator for which the corresponding dimension is exactly calculable. Only few partial results are available. For example, for the BMN-type operators [5] carrying the U(1)-charge J under one of the U(1) subgroups of the internal symmetry group the string and gauge theory expressions for the anomalous dimension appear to coincide at leading order in the large J expansion. Also, for low-energy gauge spin chain states dual to fast rotating strings the two leading coefficients in the large J, small $\frac{\lambda}{J^2}$ expansion happen to be the same on both sides of the duality [6, 7]. In the $\mathfrak{sl}(2)$ sector, few leading coefficients in both small and large λ expansions are known for the operators of the type FD^SF dual to the string spinning in AdS_5, and one may fit them approximately by a simple "square root" interpolating formula (using, e.g., the Pade approximation [8]), but this is unlikely to be the exact answer.[1]

[1] One may conjecture that it is more likely to find a hypergeometric function expression as in [9] where the dilatation operator was approximated by dropping all higher than spin-spin interaction terms.

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Recently, an interesting step towards finding the exact expressions for conformal dimensions was made: it was pointed out in [10, 11] that it is possible to obtain a closed expression for the energy of the highest energy ("antiferromagnetic") spin chain state in the $\mathfrak{su}(2)$ sector by starting with the asymptotic "gauge theory" Bethe equations of BDS [12] (that are supposed to reproduce the gauge theory results up to order λ^L in the asymptotic expansion of the large spin chain length L). The resulting expression for the highest energy $\Delta(\lambda)$ found in the $L \to \infty$ limit was given in terms of an integral of a product of two Bessel functions.[2] It has a feature expected of strong-weak coupling "interpolating" function: regular small λ expansion is smoothly connected to the $\sqrt{\lambda}$ asymptotics at large λ. Although this expression need not match the exact string theory expression, one expects [11] to find a similar expression also from the genuine quantum string Bethe ansatz (which should presumably be of the AFS "string" Bethe ansatz [14] type modified to incorporate perturbative string results). In particular, the $\sqrt{\lambda}$ strong coupling asymptotics characteristic to the large energy state is indeed found on the string theory side in the corresponding "slow-string" limit [15].

Inspired by this possibility of finding an exact expression for the conformal dimension in the $\mathfrak{su}(2)$ sector one can try to extend the work of [10, 11] by identifying similar special states in other closed subsectors of the gauge theory. In general, the spectrum of energies is unbounded in non-compact sectors but there is another special choice which is very similar to the $\mathfrak{su}(2)$ case: the so-called $\mathfrak{su}(1|1)$ sector which is the simplest sector containing gauge-invariant composite operators made of both bosonic and fermionic elementary fields of $\mathcal{N} = 4$ SYM theory.

The goal of this paper is to identify an analogue of the $\mathfrak{su}(2)$ antiferromagnetic state, i.e. the highest-energy state, for the $\mathfrak{su}(1|1)$ sector and compute its conformal dimension as a function of λ both at weak and strong coupling by starting again with the asymptotic gauge theory Bethe ansatz equations of [16, 17].

In contrast to the $\mathfrak{su}(2)$ antiferromagnetic state for which the form of the corresponding local operator ($\mathrm{tr} Z^{L/2}\Phi^{L/2} + \cdots$) is hard to describe explicitly, here the highest-dimension operator is unique and is easy to identify: it is the purely-fermionic one $\mathrm{tr}(\psi^L)$. [3] At the same time, the $\mathfrak{su}(1|1)$ gauge Bethe equations appear to be more involved than the ones describing the antiferromagnetic state in the $\mathfrak{su}(2)$ sector, making the problem of finding a closed form of their solution rather non-trivial (and remaining unsolved so far). Nevertheless, these equations appear to admit regular perturbative expansions at both small and large λ.

An obvious next question concerns realization of this highest-energy state in string theory. The conjectured "string theory analog" of the asymptotic Bethe equations [14] is

[2] The finite size corrections to the energy of the antiferromagnetic state in the $\mathfrak{su}(2)$ sector have been recently discussed in [13].

[3] In the $\mathfrak{su}(2)$ case the highest energy antiferromagnetic state has complicated structure and can be effectively described only by a density of distribution of roots in the large L limit. In contrast, the highest-energy state in the $\mathfrak{su}(1|1)$ sector is simple and the corresponding Bethe root distribution can be found also for finite L.

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known to capture some leading string energy results in certain asymptotic expansions. One is then tempted to try to find the solution of these equations (which are similar in both $\mathfrak{su}(2)$ and $\mathfrak{su}(1|1)$ sectors [14, 17]) which would correspond to a state with highest possible energy. In order to incorporate quantum string corrections [18, 19] the "string Bethe equations" of AFS [14] should undergo modifications beyond the leading order and their complete form is currently unknown. Ignoring these modifications, one may still expect [11] that the AFS-type Bethe equations should predict the same qualitative behaviour for the highest-energy state as do the gauge theory BDS equations. One implication of this is that one should change a standard $p_k \sim \frac{1}{\sqrt[4]{\lambda}}$ assumption (characteristic to the so-called short strings) about the large λ scaling behavior of momenta of elementary excitations that describe this highest-energy state at large λ: p_k should be approaching constant values at large λ. Indeed, the gauge theory ansatz predicts that the energy of this state should scale as $\sqrt{\lambda}$, while for short strings one finds $\sqrt[4]{\lambda}$ scaling law [14]. However, as we shall see below, developing a consistent strong-coupling expansion of the AFS equations in this case is not straightforward.

Let us mention also that the question about the highest-energy state illustrates the impossibility of an isolated treatment of classically-closed string sectors in quantum theory due to non-commutativity of the truncation and quantization procedures. The "reduced" $\mathfrak{su}(1|1)$ sector of string theory was described in [20] as a consistent truncation of the classical $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring equations of motion and the corresponding quantum spectrum was then found [21] by quantizing this model in the light-cone gauge where it becomes equivalent to a theory of free fermions. Its spectrum was shown to contain both short and long (winding) strings whose energies scale as $\sqrt[4]{\lambda}$ and $\sqrt{\lambda}$ respectively; there is no apparent bound on the energy since the energy of long strings can be arbitrarily increased by increasing the winding number m. What should presumably happen in the full quantum superstring treatment is that the string spectrum will become periodic in quantum numbers, so that the states with $m > L$ will be equivalent to states with $m < L$ (the same should apply also to the $\mathfrak{su}(2)$ sector case [15]).

The rest of this paper is organized as follows. In section 2 we will identify an operator in the $\mathfrak{su}(1|1)$ gauge theory sector which corresponds to the highest-energy state of the gauge theory spin chain and discuss perturbative solutions of the $\mathfrak{su}(1|1)$ BDS-type Bethe equations both at weak and strong coupling. In section 3 we shall comment on the search for a similar highest-energy state in the "string" AFS-type Bethe ansatz equations and discuss the conditions on the scaling behavior of their solution which would lead to a qualitative agreement with the gauge theory Bethe ansatz results. Finally, section 4 will contain a summary.

2. Highest-energy state from asymptotic gauge theory Bethe ansatz

The simplest $\mathcal{N} = 4$ SYM sector closed to all orders [22] is the $\mathfrak{su}(1|1)$ sector. It contains operators of the form

$$\mathrm{tr}\left(Z^{L-M}\psi^M\right) \tag{2.1}$$

with canonical dimension $\Delta_0 = L+\frac{1}{2}M$, the U(1)-charge $J = L-\frac{1}{2}M$ and the Lorentz spin $S = \frac{1}{2}M$ (Z is a complex scalar and ψ is the highest-weight component of the Weyl spinor from the vector multiplet). The integer L is identified with the length of the corresponding spin chain [23].

The state we will be interested in has $M = L$, i.e. corresponds to the operator

$$\mathrm{tr}(\psi^L)\,, \qquad\qquad \Delta = \frac{3}{2}L + \mathcal{O}(\lambda)\ . \tag{2.2}$$

This operator does not mix with other operators containing Z and thus should be an eigenstate of the dilatation operator.[4] This state provides a simple example allowing to check the consistency of solution of the analog of the BDS Bethe ansatz for the $\mathfrak{su}(1|1)$ sector suggested in [16, 17].

Let us point out the following difference with the $\mathfrak{su}(2)$ case. There one studies the operators of the type $\mathrm{tr}(Z^{J_1}\Phi^{J_2})$, where Φ is another complex scalar of the $\mathcal{N} = 4$ SYM and the length is $L = J_1 + J_2$. The highest-energy antiferromagnetic state has $J_1 = J_2$. There are *many* operators with the same charges $J_1 = J_2$ and the AF state is distinguished among them by the requirement to have the maximal energy. In the $\mathfrak{su}(1|1)$ case the state $\mathrm{tr}\psi^L$ is unique for $J = S = \frac{1}{2}L$, i.e. $M = L$, and it maximizes the energy on the space of all $\mathfrak{su}(1|1)$ operators with fixed L.

2.1 Finite L weak coupling results

Our starting point will be the all-order asymptotic Bethe ansatz for the $\mathfrak{su}(1|1)$ sector which is the analog of the $\mathfrak{su}(2)$ BDS Bethe ansatz [16, 17]

$$e^{ip_kL} = \prod_{k\neq j}^{M} \frac{1-\frac{g^2}{2x_k^+x_j^-}}{1-\frac{g^2}{2x_k^-x_j^+}}\,, \qquad\qquad g^2 = \frac{\lambda}{8\pi^2}\,. \tag{2.3}$$

Here M is the number of impurities, i.e. the number of ψ operators in (2.1). The different quantities enetring eq. (2.3) are defined as

$$p_k = \frac{1}{i}\log\frac{x_k^+}{x_k^-}\,, \qquad x_k^\pm = x^\pm(u_k)\,, \tag{2.4}$$

$$x(u) = \frac{1}{2}\Big(u+\sqrt{u^2-2g^2}\Big)\,, \qquad x^\pm = x\Big(u\pm\frac{i}{2}\Big)\ . \tag{2.5}$$

Also,

$$u(p) = \frac{1}{2}\cot\frac{p}{2}\sqrt{1+8g^2\sin^2\tfrac{p}{2}}\,, \tag{2.6}$$

$$x^\pm(p) = \frac{e^{\pm\frac{i}{2}p}}{4\sin\frac{p}{2}}\Big(1+\sqrt{1+8g^2\sin^2\tfrac{p}{2}}\Big)\,. \tag{2.7}$$

[4] Indeed, the dilatation operator is built out of products of (super)permutation operators, so $\mathrm{tr}\psi^L$ is an eigenstate just like $\mathrm{tr}Z^L$. The difference with the latter is the corresponding sign of the eigenvalue of the permutation operator (which was +1 in $\mathfrak{su}(2)$ case but is -1 in $\mathfrak{su}(1|1)$ case) and as a result $\mathrm{tr}\psi^L$ gets nontrivial anomalous dimension we aim to compute.

Taking the logarithm of (2.3) we find the following equation for p_k

$$p_k = 2\pi \frac{n_k}{L} + \frac{1}{iL} \sum_{j \neq k}^{M} \log \frac{1 - \frac{g^2}{2x_k^+ x_j^-}}{1 - \frac{g^2}{2x_k^- x_j^+}} \ , \tag{2.8}$$

where n_k are integers. The resulting dimension or energy is

$$\Delta = L + \frac{1}{2} M + E \,, \qquad E = ig^2 \sum_{k=1}^{M} \left[\frac{1}{x^+(u_k)} - \frac{1}{x^-(u_k)} \right] \tag{2.9}$$

or

$$E = \sum_{k=1}^{M} \left[\sqrt{1 + 8g^2 \sin^2 \tfrac{p_k}{2}} - 1 \right] . \tag{2.10}$$

The total world-sheet momentum

$$P = \sum_{k=1}^{M} p_k = 2\pi m \tag{2.11}$$

should be quantized due to the cyclicity of the chain (the shift operator $U = \exp(i \sum_k p_k)$ must be equal to the identity), i.e. the physical solution $\{p_k\}$ of (2.8) must be such that the "winding" m should be integer. Note that as long as m is integer, there exists an equivalent distribution $\{p_k\}$ for which m, i.e. the total momentum, vanishes (as it is usually assumed when comparing to gauge-theory operators). Indeed, both the BA equation (2.3),(2.6) and the energy (2.10) are invariant under shifts of each p_k by 2π, i.e. under

$$p_k \ \rightarrow \ p_k + 2\pi m_k \,,$$

where m_k are arbitrary integers (this is equivalent of shifting n_k in (2.8) by $m_k L$). Then $m \rightarrow m + \sum_k m_k$ and thus can be made to vanish by shifting, e.g., just one mode number.

As discussed in [16], the equation (2.8) can be solved order by order in perturbation theory in g^2. At leading 1-loop order one has simply $p_k = \frac{2\pi n_k}{L}$, $k = 1, \ldots, M$, where all n_k must be different because of Fermi statistics. We may restrict the numbers n_k to belong to a "fundamental region", e.g., $[1, L]$, or, to allow the possibility to choose $m = 0$, to $[-\frac{L-1}{2}, \frac{L-1}{2}]$.

The first two leading terms in the energy are then found to be [24, 16]

$$\begin{aligned} E = \ & 4g^2 \sum_{k=1}^{M} \sin^2 \tfrac{\pi n_k}{L} - 8g^4 \Bigg[\sum_{k=1}^{M} \sin^4 \tfrac{\pi n_k}{L} \\ & - \frac{2}{L} \sum_{k,j=1}^{M} \cos \tfrac{\pi n_k}{L} \ \sin^2 \tfrac{\pi n_k}{L} \ \sin \tfrac{\pi n_j}{L} \ \sin \tfrac{\pi (n_k - n_j)}{L} \Bigg] + \mathcal{O}(g^6) \,. \end{aligned} \tag{2.12}$$

From the 1-loop term in the energy it is clear that it is maximized when $M = L$ so that n_k take all possible distinct values from the interval $[1, L]$, i.e.

$$n_k = k \,, \qquad p_k = 2\pi \frac{k}{L} + \mathcal{O}(g^2) \,, \qquad k = 1, \ldots, L \,. \tag{2.13}$$

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Thus the maximal-energy state in the spectrum, i.e. the direct analog of the "antiferromagnetic" state in the $\mathfrak{su}(2)$ case, should correspond to the purely-fermionic operator (2.2). This unique state $\mathrm{tr}\psi^L$ is "as far as possible" from the lowest energy BPS state tr Z^L. Using (2.13) in (2.9),(2.12) we find

$$\frac{\Delta}{L} = \frac{3}{2} + 2g^2 - 4g^4 + \mathcal{O}(g^6)\,. \tag{2.14}$$

As a check of (2.12), the same expression (2.14) is found by directly applying the 1-loop and 2-loop $\mathfrak{su}(1|1)$ dilatation operator in eqs. (2.18) and (3.1) in [16] to the state $\mathrm{tr}\psi^L$ (for any finite L).[5]

Notice that we can now assume that L is an odd number: for even L the operator tr ψ^L vanishes identically (due to the clash between the cyclicity of the trace and the anticommutativity of the $\mathrm{SU}(N)$ adjoint matrix elements ψ_{AB}). Thus the winding number (2.11) for the momentum distribution (2.13)

$$m = \frac{1}{2}(L+1) \tag{2.15}$$

is indeed integer, and thus this momentum distribution is consistent with periodicity. The distribution (2.13) is indeed equivalent upon -2π shifts of upper half of the momenta to the one with n_k in the $[-\frac{L-1}{2}, \frac{L-1}{2}]$ interval:

$$p_k = 2\pi\frac{k}{L} + \mathcal{O}(g^2)\ , \qquad k = -\frac{L-1}{2}, \ldots, 0, \ldots, \frac{L-1}{2}\,. \tag{2.16}$$

By iterating the equation (2.8) with the initial condition (2.13) it is straightforward to compute few subleading terms in p_k (for finite L)[6]

$$\begin{aligned} p_k = \ & \frac{2\pi n_k}{L} - g^2 \sin\frac{2\pi n_k}{L} + \frac{g^4}{4}\left(10\sin\frac{2\pi n_k}{L} + \sin\frac{4\pi n_k}{L}\right) \\ & - \frac{g^6}{12}\left(99\sin\frac{2\pi n_k}{L} + 18\sin\frac{4\pi n_k}{L} + 4\sin\frac{6\pi n_k}{L}\right) + \mathcal{O}(g^8)\,. \end{aligned} \tag{2.17}$$

We determined several higher-order terms in this expansion but do not give them explicitly here. Then the energy of this special state with $n_k = k$ computed using (2.10) is found to be

$$\begin{aligned} \frac{\Delta}{L} &= \frac{3}{2} + 2g^2 - 4g^4 + \frac{29}{2}g^6 - \frac{259}{4}g^8 + \frac{1307}{4}g^{10} - 1790g^{12} + 10396g^{14} \\ &\quad - \frac{504397}{8}g^{16} + \frac{6324557}{16}g^{18} - \frac{40702709}{16}g^{20} + \cdots\,. \end{aligned} \tag{2.18}$$

The growth of the coefficients is an artifact of expressing Δ/L in terms of $g^2 = \frac{\lambda}{8\pi^2}$: the coefficients become numerically small when Δ is expressed in terms of λ:

$$\frac{\Delta}{L} = \frac{3}{2} + \frac{\lambda}{4\pi^2} - \frac{\lambda^2}{16\pi^4} + \frac{29\lambda^3}{1024\pi^6} - \frac{259\lambda^4}{16384\pi^8} + \frac{1307\lambda^5}{131072\pi^{10}} - \cdots\,. \tag{2.19}$$

[5] If we represent Z as spin-up state and ψ as spin-down state, then only the first $\pm(1-\sigma^3)$ terms in eqs. (2.18) and (3.1) in [16] will contribute when the dilatation operator is applied to the (01) spin-up state corresponding to $\mathrm{tr}\psi^L$.

[6] Note that $\sum_{k=1}^{L} p_k = \pi(L+1)$ does not depend on g^2.

Note that the series is sign-alternate which is consistent with a finite radius of convergence.[7] The most naive approximation to an exact expression that has finite radius of convergence and reproduces the two leading weak-coupling expansion coefficients is very simple to guess:

$$\frac{\Delta_{\rm fit}}{L} = 1 + \frac{1}{2}\sqrt{1 + \frac{\lambda}{\pi^2}}\,, \tag{2.20}$$

$$\left(\frac{\Delta_{\rm fit}}{L}\right)_{\lambda\to 0} = \frac{3}{2} + 2\frac{\lambda}{8\pi^2} - 4\frac{\lambda^2}{(8\pi^2)^2} + \frac{32}{2}\frac{\lambda^3}{(8\pi^2)^3} - \cdots\,. \tag{2.21}$$

The λ^3 coefficient here $\frac{32}{1024}$ is very close to $\frac{29}{1024}$ in (2.19).

2.2 Large L limit and equation for the Bethe root density

In a natural attempt to try to solve the equation (2.8) exactly and thus find the closed expression for the energy (2.18) let us follow the treatment of the AF state of the $\mathfrak{su}(2)$ chain in [10, 11] and consider the $L \to \infty$ limit. In this limit we can take the continuum approximation of (2.8) getting the following integral equation for the density of roots:

$$\frac{dp}{du} = -2\pi\rho(u) + \frac{1}{i}\int_{-\infty}^{\infty} dv\rho(v)\frac{\partial}{\partial u}\log\frac{1 - \frac{g^2}{2x^+(u)x^-(v)}}{1 - \frac{g^2}{2x^-(u)x^+(v)}}\,. \tag{2.22}$$

Here the density of Bethe roots $\rho(u)$ is defined as in [11] ($\xi = \frac{k}{L} \in (0,1)$)

$$\rho(u) = -\frac{d\xi}{du}\,, \qquad \int_{-\infty}^{\infty} du\ \rho(u) = 1\,, \tag{2.23}$$

and also

$$p(u) = \frac{1}{i}\log\frac{x^+(u)}{x^-(u)}\,, \qquad \frac{dp}{du} = \frac{1}{i}\left[\frac{1}{\sqrt{(u+i/2)^2 - 2g^2}} - \frac{1}{\sqrt{(u-i/2)^2 - 2g^2}}\right]. \tag{2.24}$$

$p(u)$ changes from 2π to 0 while u changes from $-\infty$ to $+\infty$. The energy shift E in (2.9) is then given by

$$\frac{E}{L} = ig^2\int_{-\infty}^{\infty} du\ \rho(u)\left[\frac{1}{x^+(u)} - \frac{1}{x^-(u)}\right]. \tag{2.25}$$

For comparison, the linear integral equation for $\rho(u)$ one finds in the $\mathfrak{su}(2)$ sector by starting with the BDS ansatz is [10, 11]

$$\frac{dp}{du} = -2\pi\rho(u) - 2\int_{-\infty}^{\infty} dv\ \rho(v)\frac{1}{(u-v)^2+1}\,. \tag{2.26}$$

Eq. (2.26) is obviously simpler than (2.22) which has less trivial kernel and thus is not readily solvable by the Fourier transform (or by using [25] the simple rule of convolution of the two kernels $K = \frac{1}{(u-v)^2+1}$ which gives a similar kernel with shifted parameters and thus allows to invert the operator $I + \frac{1}{\pi}K$). The solution of (2.26) found in [10, 11] is (J_n

[7] Inverting the sign of λ one finds that the series behaves in the qualitatively same way as $1 - \sqrt{1-x}$ expanded in x and truncated at finite order.

are Bessel functions)

$$\rho(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} ds\; e^{isu}\;\frac{\mathrm{J}_0(\frac{\sqrt{\lambda}}{\pi}s)}{2\cosh\frac{s}{2}}\,, \tag{2.27}$$

$$\frac{\Delta_{\mathfrak{su}(2)}}{L} = 1+\frac{\sqrt{\lambda}}{\pi}\int_0^{\infty}\frac{ds}{s}\;\frac{\mathrm{J}_0\left(\frac{\sqrt{\lambda}}{2\pi}s\right)\mathrm{J}_1\left(\frac{\sqrt{\lambda}}{2\pi}s\right)}{\mathrm{e}^{\,s}+1}\,. \tag{2.28}$$

While (2.22) is simply a linear integral equation, we did not find a way to solve it in a closed form. The weak-coupling perturbation theory leads to the following expression for the density

$$\rho = \sum_{k=0}^{\infty} g^{2k}\rho_k = \frac{1}{\pi}\sum_{k=0}^{\infty} g^{2k}\frac{\sum_{m=0}^{2k} a_{km}u^{2m}}{(u^2+\frac{1}{4})^{2k+1}} \tag{2.29}$$

The normalization condition $\int_{-\infty}^{\infty} du\;\rho(u)=1$ gives

$$\sum_{m=0}^{2k} 2^{-2m}\Gamma(2k-m+\tfrac{1}{2})\Gamma(m+\tfrac{1}{2})\;a_{km}=0\,. \tag{2.30}$$

Explicitly,

$$\begin{aligned}\rho(u) = {} & \frac{1}{2\pi}\frac{1}{u^2+\frac{1}{4}}+g^2\frac{-5+48u^2+16u^4}{32\pi(u^2+\frac{1}{4})^3}+\\ & +g^4\frac{-31+1060u^2-1520u^4-320u^6+512u^8}{512\pi(u^2+\frac{1}{4})^5}+\cdots\,.\end{aligned} \tag{2.31}$$

It is interesting that all the coefficients a_{km} here are integers, compared to transcendental coefficients in the $\mathfrak{su}(2)$ case (suggesting that a closed form of the solution may be simpler than (2.27)). Substituting (2.31) (with proper number of higher-order terms included) into the expression for the energy (2.9),(2.25) we find the same result (2.18) as obtained for finite L.

2.3 Strong coupling expansion

Let us now try to solve the Bethe equation (2.8) or its large L version (2.22) in the strong coupling limit of $g\sim\lambda\gg 1$. It is useful to switch to momentum representation $u\to p$ (see (2.6)). We shall assume that p_k admit the following expansion

$$p_k = p_k^{(0)}+\frac{p_k^{(1)}}{\sqrt{\lambda}}+\cdots\,. \tag{2.32}$$

At strong coupling (2.7) gives (we omit the label $^{(0)}$ on p_k for notational simplicity)

$$x^{\pm}\to\frac{\sqrt{\lambda}}{4\pi}\;e^{\pm\frac{i}{2}p}\epsilon(p)+\cdots\,,\qquad \epsilon(p)\equiv\mathrm{sign}\left(\sin\frac{p}{2}\right)\,,\qquad \epsilon(0)=0\,. \tag{2.33}$$

Then the strong coupling limit of the Bethe equations (2.3) becomes

$$e^{ip_kL}=\prod_{j\neq k}^{M}\frac{1-e^{-\frac{i}{2}(p_k-p_j)}\epsilon(p_k)\epsilon(p_j)}{1-e^{\frac{i}{2}(p_k-p_j)}\epsilon(p_k)\epsilon(p_j)}+\mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right). \tag{2.34}$$

For $M=L$=odd these equations can be solved as follows. Let assume that there exists a solution with all $\epsilon(p_k)=1$ (this does not restrict the generality: if for some p_k we have $\epsilon(p_k)=-1$ we can shift it by 2π and change this sign). Then

$$e^{ip_kL}=\prod_{j\neq k}^{L}e^{i\pi-\frac{i}{2}(p_k-p_j)}+\mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right), \tag{2.35}$$

and thus (since $(e^{i\pi})^{L-1}=1$ for L odd)

$$\frac{3}{2}p_kL=2\pi n_k+\pi m+\mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right),\qquad m\equiv\frac{1}{2\pi}\sum_{j=1}^{L}p_j\,. \tag{2.36}$$

From here $m=\frac{1}{L}\sum_{k=1}^{L}n_k$, and this number must be integer. Assuming, as at weak coupling, that $n_k=k=1,\dots,L$ we get $m=\frac{1}{2}(L+1)$ which is indeed integer for odd L. As a result, we find a consistent solution for momentum distribution at strong coupling

$$p_k=\frac{4\pi k}{3L}+\frac{\pi}{3}\frac{L+1}{L}\,,\qquad k=1,\dots,L\,. \tag{2.37}$$

Here all p_k lie on the interval $(0,2\pi)$ so with this choice one has indeed $\mathrm{sign}(\sin\frac{p_k}{2})=1$. One can also choose an equivalent distribution with $m=0$ by shifting one or few momenta by multiples of 2π. For example, for $L=3$ a choice with $m=0$ is $p_k=(-\frac{20\pi}{9},\frac{8\pi}{9},\frac{12\pi}{9})$.

At the subleading $\frac{1}{\sqrt{\lambda}}$ order we find that the logarithm of the Bethe equations takes the form

$$p_k^{(1)}=\sum_{j\neq k}^{L}\left[\frac{p_j^{(1)}-p_k^{(1)}}{2L}+\frac{\pi}{L}\frac{\left(\frac{\epsilon(p_k)}{\sin\frac{p_k}{2}}+\frac{\epsilon(p_j)}{\sin\frac{p_j}{2}}\right)\sin\frac{p_j-p_k}{2}}{1-\cos\frac{p_j-p_k}{2}\epsilon(p_j)\epsilon(p_k)}\right],\qquad p_k\equiv p_k^{(0)} \tag{2.38}$$

and it allows one to determine the correction to the leading momenta (2.37). If we assume that $\sum_k^L p_k^{(1)}=0$ then we obtain

$$p_k^{(1)}=\frac{2\pi}{3L}\sum_{j\neq k}^{L}\frac{\left(\frac{\epsilon(p_k)}{\sin\frac{p_k}{2}}+\frac{\epsilon(p_j)}{\sin\frac{p_j}{2}}\right)\sin\frac{p_j-p_k}{2}}{1-\cos\frac{p_j-p_k}{2}\epsilon(p_j)\epsilon(p_k)}\,. \tag{2.39}$$

Since all $\epsilon(p_k)=1$, then finally

$$p_k^{(1)}=\frac{2\pi}{3L}\sum_{j\neq k}^{L}\cot\frac{p_j-p_k}{4}\left(\frac{1}{\sin\frac{p_k}{2}}+\frac{1}{\sin\frac{p_j}{2}}\right), \tag{2.40}$$

where indeed $\sum_k^L p_k^{(1)}=0$.

The strong-coupling expansion of the energy (2.10) is

$$E = \frac{\sqrt{\lambda}}{\pi}\sum_{k=1}^{L}\left|\sin\frac{p_k}{2}\right| - L + \frac{1}{2\pi}\sum_{k=1}^{L} p_k^{(1)}\cos\frac{p_k}{2}\,\epsilon(p_k) + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right). \tag{2.41}$$

In the present case we find at large L

$$\frac{\Delta}{L} = \frac{E}{L}+\frac{3}{2} = c_1\sqrt{\lambda}+c_2+\mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right), \tag{2.42}$$

$$(c_1)_{L\to\infty} = \frac{3\sqrt{3}}{2\pi^2}\approx 0.26\,, \tag{2.43}$$

$$(c_2)_{L\to\infty} = \frac{1}{2}+\left[\frac{1}{3L^2}\sum_{k,j=1}^{L}\cot\frac{p_j-p_k}{4}\left(\frac{1}{\sin\frac{p_k}{2}}+\frac{1}{\sin\frac{p_j}{2}}\right)\cos\frac{p_k}{2}\right]_{L\to\infty}\approx 1.18\,, \tag{2.44}$$

where to compute $(c_2)_{L\to\infty}$ we used (2.37) and numerically evaluated the sum.

For comparison, in the $\mathfrak{su}(2)$ case by starting from the exact solution at infinite L and weak coupling λ (2.28) and interpolating to strong coupling one finds

$$\left(\frac{\Delta_{\mathfrak{su}(2)}}{L}\right)_{L\to\infty,\ \lambda\to\infty} = \frac{\sqrt{\lambda}}{\pi^2}+\frac{3}{4}+\cdots\,, \tag{2.45}$$

where dots stand for exponentially small corrections.[8] This seems to suggest that the strong-coupling expansion of the solution of the BDS-type Bethe ansatz may turn out to be only asymptotic also in other sectors.[9] In the absence of a closed expression for the energy, i.e. the $\mathfrak{su}(1|1)$ counterpart of (2.28) in the $\mathfrak{su}(2)$ sector, we are unable to decide if the expansion in (2.42) will or will not contain exponential corrections.[10] Still, it is amusing to note that the most naive interpolation formula (2.20) that reproduced exactly the first two leading weak-coupling expansion coefficients gives also a relatively good fit at strong coupling:

$$\left(\frac{\Delta_{\rm fit}}{L}\right)_{L\to\infty,\ \lambda\to\infty} = \frac{\sqrt{\lambda}}{2\pi}+1+\mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right) = 0.16\sqrt{\lambda}+1+\mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right), \tag{2.46}$$

where the coefficients 0.16 and 1 are not that far from 0.26 and 1.18 in (2.44).

One may question if the above strong coupling solution for the distribution of p_k is unique.[11] It may seem indeed that for finite L one may find many similar solutions with different momentum range. However, one should remember that the asymptotic BDS-type

[8]We thank A. Tirziu and K. Zarembo for a discussion of this expansion.

[9]We thank M. Staudacher for suggesting this to us.

[10]These exponential corrections are likely to be an artifact of the BDS ansatz related to the order of limits problem and might be absent on the string theory side where there is no an apparent reason for $e^{-\frac{1}{\sqrt{\lambda}}}$-terms (no world-sheet instantons, etc). In general, the interpolation from weak coupling (gauge-perturbative region) to strong coupling (string perturbative region) should be done in the full expression for the energy $E(\lambda, L)$ which need not be the same as the one coming out of the asymptotic BDS equations.

[11]We thank D. Serban for stressing this issue.

ansatz is related to gauge theory only to order λ^L, i.e. keeping L finite while taking λ large may not be consistent. One might further expect that different solutions which admit regular large L limit will, in fact, be equivalent, i.e. will lead *in this limit* to the same expression for the energy. To illustrate this point, let us mention that there exists another solution of the strong coupling Bethe equations (2.34) which has momenta p_k symmetrically distributed around zero

$$\begin{aligned} p_k &= \frac{4\pi k}{3L-1} + \pi\frac{L-1}{3L-1}, \qquad && k = 1, \ldots, (L-1)/2 \\ p_0 &= 0\,, && \\ p_k &= \frac{4\pi k}{3L-1} - \pi\frac{L-1}{3L-1}, && k = -(L-1)/2, \ldots, -1\,. \end{aligned} \tag{2.47}$$

For a symmetric distribution the Bethe equations (2.34) take the form

$$e^{ip_k L} = \prod_{j=1}^{\frac{L-1}{2}} \frac{1 - e^{-\frac{i}{2}(p_k - p_j)}}{1 - e^{\frac{i}{2}(p_k - p_j)}} \frac{1 + e^{-\frac{i}{2}(p_k + p_j)}}{1 + e^{\frac{i}{2}(p_k + p_j)}}, \qquad k = 1, \ldots, (L-1)/2\,, \tag{2.48}$$

which are indeed solved by (2.47). In the large L limit the momenta spread over the interval $(-\pi, -\frac{\pi}{3}) \cup (\frac{\pi}{3}, \pi)$. The energy of this solution in the large L limit has the *same* leading term, eq.(2.43), the subleading terms are however different. Without further input one can not decide if the solutions we found indeed correspond to the highest-energy state as it is seen from the strong coupling perspective.

The same result for the leading strong-coupling term in the energy (2.42) can be found also from the strong-coupling limit of the integral equation (2.22) after converting it into the "momentum" form:

$$1 = 2\pi\tilde{\rho}(p) + \frac{1}{i}\int dq\ \tilde{\rho}(q)\frac{\partial}{\partial p}\log\frac{1 - \frac{\lambda}{(4\pi)^2 x^+(p) x^-(q)}}{1 - \frac{\lambda}{(4\pi)^2 x^-(p) x^+(q)}}\,. \tag{2.49}$$

Here the momentum density is (recall that $\frac{du}{dp} < 0$)

$$\tilde{\rho}(p) \equiv -\frac{du}{dp}\rho(u)\ , \qquad \int_{p_{min}}^{p_{max}} dp\ \tilde{\rho}(p) = 1\,, \tag{2.50}$$

and the limiting values p_{max}, p_{min} may, in general, depend on λ. Using (2.33) we get (assuming that within (p_{max}, p_{min}) $\sin\frac{p}{2}$ has positive sign)

$$1 = 2\pi\ \tilde{\rho}(p) + \frac{1}{i}\int_{q_{min}}^{q_{max}} dq\ \tilde{\rho}(q)\frac{\partial}{\partial p}\log\frac{1 - e^{-\frac{i}{2}(p-q)}}{1 - e^{\frac{i}{2}(p-q)}} + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)\,. \tag{2.51}$$

From here $1 = 2\pi\ \tilde{\rho}(p) - \frac{1}{2} + \mathcal{O}(\frac{1}{\sqrt{\lambda}})$, i.e.

$$\tilde{\rho}(p) = \frac{3}{4\pi} + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)\,. \tag{2.52}$$

Thus the momentum density is constant as it was at weak coupling ($\tilde{\rho}(p) = \frac{1}{2\pi}$, see (2.17)), but the momentum distribution range have changed from 2π to $\frac{4\pi}{3}$. This is exactly what we have found above in the discrete (finite L) approach (2.37). To match (2.37) we are to choose (at $L \to \infty$) $p_{min} = \frac{\pi}{3}$, $p_{max} = \frac{5\pi}{3}$.[12]

Similar observations were made in the $\mathfrak{su}(2)$ sector in [11], where the momentum quantization condition and thus the coefficient in the momentum density had changed by the factor of 2 in going from the weak to strong coupling.[13]

The general expression for the energy in the continuum limit in the momentum form is (see (2.9),(2.25))

$$\frac{\Delta}{L} = \frac{1}{2} + \int_{p_{min}}^{p_{max}} dp\ \tilde{\rho}(p)\ \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2}}\,, \tag{2.53}$$

so that at strong coupling we get the same result as in (2.42),(2.43)

$$\left(\frac{\Delta}{L}\right)_{\lambda\to\infty} = \frac{\sqrt{\lambda}}{\pi} \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} dp\ \tilde{\rho}(p) \left|\sin \frac{p}{2}\right| + \cdots = \frac{3\sqrt{3}}{2\pi^2} \sqrt{\lambda} + \cdots\,, \tag{2.54}$$

where we used (2.52).

3. Remarks on highest-energy state in the spectrum of the "string" Bethe ansatz

In [14] a novel type of the Bethe ansatz equations was introduced to describe the leading quantum corrections to the spectrum of classical strings on $AdS_5 \times S^5$. Originally conjectured for the $\mathfrak{su}(2)$ sector, this ansatz was subsequently generalized to other sectors of rank one [16], and finally to the full string sigma model on $AdS_5 \times S^5$ [17].

If $s = -1, 0, 1$ for the $\mathfrak{sl}(2)$, $\mathfrak{su}(1|1)$ and $\mathfrak{su}(2)$ sectors then the conjectured "quantum string" Bethe ansatz equations can be written in the form [17]

$$e^{ip_kL} = \prod_{k\neq j}^{M} \left(\frac{x_k^+ - x_j^-}{x_k^- - x_j^+}\right)^s \frac{1 - \frac{g^2}{2x_k^+ x_j^-}}{1 - \frac{g^2}{2x_k^- x_j^+}}\ e^{i\theta(p_j,p_k)}\,, \tag{3.1}$$

$$e^{i\theta(p_j,p_k)} = \left(\frac{1 - \frac{g^2}{2x_k^+ x_j^-}}{1 - \frac{g^2}{2x_k^- x_j^+}}\right)^{-2} \left(\frac{1 - \frac{g^2}{2x_k^- x_j^+}\ \ 1 - \frac{g^2}{2x_k^+ x_j^-}}{1 - \frac{g^2}{2x_k^+ x_j^+}\ \ 1 - \frac{g^2}{2x_k^- x_j^-}}\right)^{2i(u_k-u_j)}\,, \tag{3.2}$$

where the definitions of $x_k^\pm, u_k$ are the same as in (2.3)–(2.7) and $e^{i\theta(p_j,p_k)}$ is an extra "string" S-matrix factor that distinguishes the string BA from asymptotic gauge BA (indeed, after omitting this factor and setting $s = 0$ eq.(3.1) reduces to (2.3)).

[12] The momentum interval is fixed so that its length is $\frac{4\pi}{3}$ to have $\tilde{\rho}(p)$ normalized and also to satisfy the assumption that $\sin \frac{p}{2}$ has positive sign.

[13] In the $\mathfrak{su}(2)$ sector one can get (from (2.27)) a closed formula for $\tilde{\rho}(p)$ as a function of λ that interpolates between the two constant values at $\lambda = 0$ and $\lambda = \infty$.

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In the $\mathfrak{su}(1|1)$ sector we are interested in here the logarithm of the string Bethe equations (3.1) reads

$$p_k = 2\pi \frac{n_k}{L} + \frac{1}{iL} \sum_{j \neq k}^{M} \left(\log \frac{1 - \frac{g^2}{2x_k^- x_j^+}}{1 - \frac{g^2}{2x_k^+ x_j^-}} + 2i(u_k - u_j) \log \frac{1 - \frac{g^2}{2x_k^- x_j^+}}{1 - \frac{g^2}{2x_k^+ x_j^+}} \frac{1 - \frac{g^2}{2x_k^+ x_j^-}}{1 - \frac{g^2}{2x_k^- x_j^-}} \right) . \tag{3.3}$$

Here n_k are the excitation numbers and $L = J + \frac{1}{2}M$. As soon as the momenta p_k solving (3.3) are found, the energy can be computed by using the formula (2.10).

3.1 Weak coupling expansion

While the equations (3.3) were originally found by "discretising" the integral equations which appear in the semiclassical string theory where g is large [7], they also admit a regular weak-coupling limit $g \to 0$ [26]. Assuming the same distribution of the numbers n_k as in the weak-coupling gauge theory (2.13), one can solve equations (3.3) perturbatively. In particular, few leading terms of the momentum p_k read

$$\begin{aligned} p_k = \; & \frac{2\pi n_k}{L} - g^2 \sin \frac{2\pi n_k}{L} + \frac{1}{4} g^4 \left(2 \sin \frac{2\pi n_k}{L} + 5 \sin \frac{4\pi n_k}{L} \right) \\ & - \frac{1}{12} g^6 \left(-23 \sin \frac{2\pi n_k}{L} + 64 \sin \frac{4\pi n_k}{L} + 10 \sin \frac{6\pi n_k}{L} \right) + \mathcal{O}(g^8) , \end{aligned} \tag{3.4}$$

This leads to the following expansion for the energy (2.10)

$$\frac{\Delta}{L} = \frac{3}{2} + 2g^2 - 4g^4 + \frac{25}{2} g^6 - \frac{601}{12} g^8 + \frac{2849}{12} g^{10} + \cdots \tag{3.5}$$

eea As expected, the first two (one-loop g^2 and two-loop g^4) coefficients are the same as in (2.18) but the two series differ starting with g^6. Note, however, that again the series is sign-alternate and should have a finite radius of convergence. Compared to the gauge Bethe ansatz equations of the previous section, here it seems even more challenging to try to find the solution of the equations (3.3) in a closed form.

Let us recall that the equations (3.3) are known to receive the $1/g \sim 1/\sqrt{\lambda}$ corrections [19, 18] (required in order to reproduce quantum string results) which could be universally incorporated [18] in the infinite set of functions $c_r(\lambda)$. These functions define a more general *interpolating* string Bethe ansatz [14] and should lead to a modification of the weak coupling expansion (3.5) which, hopefully, should agree with that found on the gauge theory side in the large L limit.

3.2 Strong coupling expansion

Since we are interested in the highest-energy state with $M = L$ impurities, the expression for the energy (2.10) suggests that the maximal energy would be attained if all L momenta p_k were equal to π (modulo $2\pi n$. Then all $\sin \frac{p_k}{2} = 1$ and at strong coupling $E \to \frac{\sqrt{\lambda}}{\pi} L$. However, all momenta must be distinct since otherwise the wave function vanishes due to Fermi statistics of excitations, i.e. one is to choose some non-trivial distribution for p_k. As a result, the coefficient $\frac{E}{\sqrt{\lambda} L}$ will be less than 1 (it was $\frac{3\sqrt{3}}{2\pi} \approx 0.83$ in the gauge Bethe ansatz case (2.42)).

We expect the energy (2.10) to scale as $\sqrt{\lambda}L$, so we should assume that the leading term in the strong-coupling expansion of momenta should be constant, i.e. as in (2.32), $p_k = p_k^{(0)} + \frac{p_k^{(1)}}{\sqrt{\lambda}} + \cdots$,[14] where $p_k^{(0)}$ should be again distributed, say on $(-\pi, \pi)$.

Extracting this distribution from the strong coupling expansion of the string Bethe ansatz (3.3) appears to be more subtle than in the gauge Bethe ansatz case of section 2.3. Observing that according to (2.6)

$$u(p)_{\lambda\to\infty} \;\;\to\;\; \frac{\sqrt{\lambda}}{2\pi}\epsilon(p)\cos\frac{p}{2}\,, \tag{3.6}$$

where $\epsilon(p)$ is the sign factor defined in (2.33), we find that in the limit when $\lambda \to \infty$ the term proportional to $u_k - u_j$ in the eq.(3.3) provides a dominant contribution

$$\frac{i\sqrt{\lambda}}{\pi}\sum_{j\neq k}^{M}\Big[\epsilon(p_k)\cos\frac{p_k}{2} - \epsilon(p_j)\cos\frac{p_j}{2}\Big]\log\frac{1-\epsilon(p_k)\epsilon(p_j)\cos\frac{p_k-p_j}{2}}{1-\epsilon(p_k)\epsilon(p_j)\cos\frac{p_k+p_j}{2}}\,, \tag{3.7}$$

where again for simplicity we renamed $p_k^{(0)} \to p_k$. Setting $M = L$ and assuming that there exists a solution with all $\epsilon(p_k) = 1$ we obtain the following non-linear equations for the momentum distribution

$$\sum_{j\neq k}^{L}\Big(\cos\frac{p_k}{2} - \cos\frac{p_j}{2}\Big)\log\frac{\sin^2\frac{p_k-p_j}{4}}{\sin^2\frac{p_k+p_j}{4}} = 0\,. \tag{3.8}$$

It is unclear at the moment how to find a solution of this set of non-linear equations assuming that p_k obey an additional constraint $\epsilon(p_k) = 1$. Moreover, it is also unclear if the resulting expansion around this solution will be regular. Nevertheless, once a solution to eq.(3.8) is found, the energy of this state is guaranteed to have the same $\sqrt{\lambda}$ scaling behavior as found in the strong-coupling gauge theory. We also note that treating $p_k \equiv x_k$ as positions on M particles on a circle of length the 2π, eqs.(3.8) can be thought of as equations determining an equilibrium configuration $\frac{\partial U}{\partial x_k} = 0$ for some potential U. It would be important to develop this interpretation further, in particular, to see whether solutions of eq.(3.8) can be related to zeros of some known orthogonal polynomials.

It is interesting to note that the strong-coupling equations (3.7) appear to be universal: they are found from (3.1) for any value of the power s. Hence exactly the same problem appears in determining the maximal-energy state as described by the string Bethe ansatz also in the $\mathfrak{su}(2)$ sector. Moreover, the leading term in the strong-coupling expansion of the energy will then be the same as in the $\mathfrak{su}(1|1)$ and $\mathfrak{su}(2)$ sectors.[15]

Let us mention also the large L form of the above Bethe equations. In the momentum representation of the string Bethe ansatz equations the analog of (2.49) becomes

$$1 = 2\pi\tilde{\rho}(p) + \frac{1}{i}\int dq\;\tilde{\rho}(q)\frac{\partial}{\partial p}\tilde{K}(p,q)\,, \tag{3.9}$$

[14] As was discussed in [14], for short strings a natural expansion of momenta is $p_k = \frac{p_k^{(0)}}{\lambda^{1/4}} + \frac{p_k^{(1)}}{\lambda^{1/2}} + \cdots$ leading to the $\sqrt[4]{\lambda}$ scaling of the energy of the corresponding states.

[15] A similar universality is found in the spectrum of short strings in the strong-coupling ("flat-space") limit.

where $\tilde{K}(p,q) \equiv K(u(p),v(q))$ is found by using (3.3),(2.7). Taking the strong-coupling limit *assuming* that p is fixed in this limit, we get

$$\tilde{K}(p,q) = \sqrt{\lambda}\tilde{K}_1(p,q) + \tilde{K}_2(p,q)\,, \tag{3.10}$$

where

$$\tilde{K}_2 = \log\frac{1-e^{\frac{i}{2}(p-q)}\epsilon(p)\epsilon(q)}{1-e^{-\frac{i}{2}(p-q)}\epsilon(p)\epsilon(q)} = \frac{i}{2}(p-q) + \mathrm{const}\,, \tag{3.11}$$

$$\tilde{K}_1 = \frac{i}{\pi}\left[\epsilon(p)\cos\frac{p}{2} - \epsilon(q)\cos\frac{q}{2}\right]\log\left[\frac{1-e^{\frac{i}{2}(p-q)}\epsilon(p)\epsilon(q)}{1-e^{-\frac{i}{2}(p+q)}\epsilon(p)\epsilon(q)}\frac{1-e^{-\frac{i}{2}(p-q)}\epsilon(p)\epsilon(q)}{1-e^{\frac{i}{2}(p+q)}\epsilon(p)\epsilon(q)}\right]. \tag{3.12}$$

If as in the gauge BA case the leading term in $\tilde{\rho}(p)$ for the highest-energy state does not depend on λ, we need to ensure that the leading term $\tilde{K}_1$ does not contribute to (3.9). Assuming that for our solution $\tilde{\rho}$=const we are to satisfy

$$\int dq\,\frac{\partial}{\partial p}\tilde{K}_1(p,q) = 0 \tag{3.13}$$

which is the continuum analog of the vanishing of (3.7). The same discussion of the leading large λ asymptotics applies also to the $\mathfrak{su}(2)$ case. It is not clear how to satisfy the condition (3.13), and this may be indicating a potential problem in direct application of the AFS-type Bethe ansatz to determining the highest-energy state at strong coupling.

3.3 Comments on the spectrum of reduced $\mathfrak{su}(1|1)$ string model

In the absence of direct information about the structure of exact string spectrum one may try to draw some lessons from "reduced" models obtained by truncating the string degrees of freedom at the classical level and then quantizing the remaining modes. While the truncation and quantization procedures are not expected to commute, the spectrum of reduced model may still reflect certain features of the exact string spectrum. Let us finish this section with a review of the structure of the spectrum of the reduced $\mathfrak{su}(1|1)$ model [20, 21].

At the classical level the $\mathfrak{su}(1|1)$ sector of the $\mathrm{AdS}_5\times\mathrm{S}^5$ string theory can be defined as a consistent truncation of the superstring equations of motion [20]. In the light-cone gauge this model reduces to the theory of a free massive world-sheet Dirac fermion, and therefore, can be easily quantized [21]. The corresponding spectrum correctly reproduces the leading $1/J$-corrections to the energy of the plane-wave states but at higher orders leads to the results which are different from the ones predicted by the "string" Bethe equations (3.3). [16] Assuming that the AFS ansatz does in fact represent the correct quantum string

[16]Truncating the classical superstring equations of motion in the *temporal* gauge one finds a new non-linear classically-integrable fermionic model [20]. This non-linear AAF model, however, is not power-counting renormalizable at the quantum level and is not readily solvable (though its low-energy 2-body S-matrix and thus the corresponding Bethe ansatz may be computed assuming quantum integrability [27]). While at the classical level the light-cone gauge and temporal gauge reduced models are equivalent, this is not so at the quantum level. The difference does not appear at order $1/J$ (the near plane-wave limit) but

spectrum, one natural interpretation of this difference is that beyond the leading order the string modes which were truncated away at the classical level start to contribute. Given that the quantum version of the reduced $\mathfrak{su}(1|1)$ model leads to an approximate description of the AFS-type Bethe equations in the $\mathfrak{su}(1|1)$ sector in the region of small momenta p_k it is instructive to review the scaling behavior of the energy of different states in this model.

The momenta p_k of the elementary excitations in the reduced $\mathfrak{su}(1|1)$ model are subject to the following Bethe-type equations [21] (see also [16])

$$Jp_k = 2\pi n_k + \frac{1}{2}\sum_{j\neq k}^{M}\left(p_j\sqrt{1+\frac{\lambda p_k^2}{4\pi^2}} - p_k\sqrt{1+\frac{\lambda p_j^2}{4\pi^2}}\right), \tag{3.14}$$

where the expression under the sum is the logarithm of the two-body "string S-matrix" and n_k are integers. The energy of the corresponding state is

$$E = J + \sum_{j=1}^{M}\sqrt{1+\frac{\lambda p_j^2}{4\pi^2}}\,. \tag{3.15}$$

The spectrum contains two types of string excitations: short strings with vanishing winding $m = \sum_{k=1}^{M} p_k$ and long strings with $m \neq 0$. For short strings the energy scales in the large λ limit as $\sqrt[4]{\lambda}$ and this scaling is perfectly consistent with one predicted by the AFS Bethe equations.

For long strings the situation is different. Summing up eqs.(3.14) we get a condition

$$J\sum_{k=1}^{M} p_k = 2\pi\sum_{k=1}^{M} n_k \qquad \Longrightarrow \qquad Jm = \sum_{k=1}^{M} n_k\,. \tag{3.16}$$

Thus, for non-vanishing winding we have to assume that the momenta have the following expansion[17]

$$p_k = p_k^{(0)} + \frac{p_k^{(1)}}{\sqrt{\lambda}} + \cdots\,, \tag{3.17}$$

where the leading term $p_k^{(0)}$ is constant. Then in large λ limit only the second "string S-matrix" term in (3.14) matters. Expanding eq.(3.14) we find at the two leading orders the following equations

$$\sqrt{\lambda}: \qquad \sum_{j\neq k}(p_j^{(0)}|p_k^{(0)}| - p_k^{(0)}|p_j^{(0)}|) = 0\,, \tag{3.18}$$

the two models start to disagree at order $1/J^2$ where the AAF model first gets nontrivial UV divergencies and where the omitted interactions with other sectors (which cancel the divergences in the full superstring theory) become important. That means, in particular, that from the string theory perspective one cannot trust the non-linear AAF model more than the free light-cone gauge model beyond the $1/J$ level.

[17] Here we assume that all $p_k^{(0)}$ are non-zero. One can consider a possibility that only a part of $p_k^{(0)}$ is non-zero. This will lead to modifications of the expansions discussed below.

$$\lambda^0: \qquad Jp_k^{(0)} - 2\pi n_k - \frac{1}{4\pi}\sum_{j\neq k}\left(p_k^{(0)}p_j^{(1)} + p_j^{(0)}p_k^{(1)}\right)\left(\mathrm{sign}(p_k^{(0)}) - \mathrm{sign}(p_j^{(0)})\right) = 0\,.$$

The first equation implies

$$\sum_{j=1}^{M}|p_j^{(0)}| = m\frac{|p_k|^{(0)}}{p_k^{(0)}} = m\ \mathrm{sign}p_k^{(0)} \quad \text{for any } k.$$

Thus $p_k^{(0)}$ must be either all positive or all negative. Assuming that they are all positive we conclude from the second equation that $p_k^{(0)} = \frac{2\pi n_k}{J}$, where all $n_k > 0$. It is rather interesting that the leading equation (3.18) is satisfied identically provided $p_k^{(0)}$ are all positive or all negative. One can go further and find that with our assumption of positivity of n_k the next order in the expansion of the Bethe equations (3.14) leads to the determination of $p_k^{(1)}$:

$$\frac{1}{\sqrt{\lambda}}: \qquad p_k^{(1)} = \frac{\pi}{2J}\sum_{k\neq j}^{M}\left(\frac{n_j}{n_k} - \frac{n_k}{n_j}\right). \tag{3.19}$$

Thus, the strong coupling expansion of a long string configuration is well-defined and the leading momenta are

$$p_k = \frac{2\pi n_k}{J} + \frac{\pi}{2J\sqrt{\lambda}}\sum_{k\neq j}^{M}\left(\frac{n_j}{n_k} - \frac{n_k}{n_j}\right) + \cdots\,. \tag{3.20}$$

The expansion for the energy is therefore

$$E = \frac{\sqrt{\lambda}}{J}\sum_{k=1}^{M} n_k + J + \frac{J}{2\sqrt{\lambda}}\sum_{k=1}^{M}\frac{1}{n_k} - \frac{J}{16\lambda}\sum_{k,j=1}^{M}\frac{(n_k^2 - n_j^2)^2}{n_k^3 n_j^3} + \cdots\,. \tag{3.21}$$

Thus in the case of non-trivial winding (long strings) the energy scales as $\sqrt{\lambda}$ [21]. Here the leading term in the energy scales as $E = \frac{\sqrt{\lambda}}{J}\sum_k^M n_k$, i.e. as $E = \sqrt{\lambda}m$ if we use (3.16).[18] However, the energy is not bounded from above: the string excitation numbers n_k can be arbitrarily large making the winding and energy unbounded. One might expect that true quantum string states will develop periodicity in n_k so that m will be bounded from above by a number of order J, and thus there will exist a maximal-energy state.

Indeed, one may speculate that the non-trivial dependence of the reduced model energy on the winding number m in this simplified model is an artifact of the perturbative expansion while in the full quantum string theory the explicit dependence on m will be traded for a periodic dependence on n_k. Eq.(3.16) shows that the winding is not an independent variable but can be expressed in terms of the excitation numbers n_k. If the exact dispersion relation of the quantum string theory will indeed appear to be periodic, i.e. invariant under the shift $p_k \to p_k + 2\pi$ ($n_k \to n_k + J$) then one will always be able

[18]The $\sqrt{\lambda}m$ behavior of energy of long wound strings was observed earlier in more general context in [28].

	Gauge BA	String BA
Weak coupling	$p_k = \frac{2\pi k}{L} + \mathcal{O}(\lambda)$	$p_k = \frac{2\pi k}{L} + \mathcal{O}(\lambda)$
Strong coupling	$p_k = \frac{4\pi k}{3L} + \frac{\pi}{3}(1 + \frac{1}{L}) + \mathcal{O}\Big(\frac{1}{\sqrt{\lambda}}\Big)$	$p_k = p_k^{(0)}(L) + \mathcal{O}\Big(\frac{1}{\sqrt{\lambda}}\Big)$

Table 1: Leading momentum behavior of maximal-energy solution of gauge and string Bethe ansatze in the $\mathfrak{su}(1|1)$ sector.

to choose a momentum distribution which has zero winding without changing the value of the energy which will then be bounded from above due to the compactness of the phase space.

In spite of the fact that the reduced model does not appear to describe exact quantum string states it exhibits the following features which we expect to find in the genuine quantum string theory: (i) it suggests that the energy can indeed have $\sqrt{\lambda}$ behavior at large λ, and (ii) to get the $\sqrt{\lambda}$ scaling of the energy, the momenta of the corresponding elementary excitations should have the same large λ expansion as in eq. (3.17). These were the properties which were implemented in the solution of the gauge Bethe ansatz equations in section 2 and also the ones we were assuming above in trying to solve the AFS-type string Bethe equations. In fact, the expansion of (3.14) is very similar to the expansion of the string Bethe equations (3.3) in the strong coupling limit (cf. (3.18) and (3.7),(3.8)).

4. Summary

The maximal energy state in the $\mathfrak{su}(1|1)$ sector we discussed in this paper is special: this is one of very few cases when we know explicitly the exact quantum operator as well the corresponding distribution of momenta describing it as a solution of the gauge/string Bethe ansatz equations.

Let us summarize the asymptotic expansions for the momentum distributions we have found above (see table 1).

We have argued that the same qualitative features of the maximal energy state as found from the gauge theory BDS-type gauge Bethe ansatz should appear also for the associated quantum string state but we were unable to solve the strong-coupling limit of the conjectured AFS-type string Bethe equations explicitly. In fact, the strong-coupling behavior of the corresponding integral equation kernels in these "gauge" and "string" Bethe ansatze appears to be very different. This may be seen as another indication of a need for clarifying the structure of the "string" Bethe ansatz and, in particular, for understanding whether and how it captures the higher-energy tail of the string spectrum.

Note added: while this paper was prepared for publication we learned about a very closely related unpublished work by D. Serban and M. Staudacher [29] who also found

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the perturbative solutions of the asymptotic gauge and string Bethe ansatze in the $\mathfrak{su}(1|1)$ sector for the purely fermionic state (in particular, the expressions in eqs. (2.18), (2.31), (3.5)).

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